Space Curve Physics

Goal: Give physical interpretation to notions in space curves.

Mathese	Interpretation
$\vec{r}(t)$	position
$\vec{r}\,'(t)$	velocity
$\lvert\vec{r}\,'(t)\rvert$	speed
$\vec{r}\,''(t)$	acceleration

Ex: Compute the velocity, acceleration, and speed of a particle with position function $\vec{r}(t) = \langle \ln(t), \sqrt{t^2+1}, e^{2t} \rangle$.

Sol:

Velocity:

$$\vec{v}(t) = \vec{r}\,'(t)$$

$$= \frac{d}{dt}\left[\langle \ln(t), (t^2+1)^{1/2}, e^{2t}\rangle\right]$$

$$= \left\langle \frac{d}{dt}[\ln(t)], \frac{d}{dt}\left[(t^2+1)^{1/2}\right], \frac{d}{dt}[e^{2t}] \right\rangle$$

$$= \left\langle \frac{1}{t}, \frac{1}{2}\cdot(t^2+1)^{-\frac{1}{2}}\cdot 2t, 2e^{2t} \right\rangle$$

$$= \left\langle \frac{1}{t}, t(t^2+1)^{-1/2}, 2e^{2t} \right\rangle = \left\langle t^{-1}, t(t^2+1)^{-1/2}, 2e^{2t} \right\rangle$$

speed:

$$|\vec{v}(t)| = \sqrt{\left(\frac{1}{t}\right)^2 + \left(t(t^2+1)^{-1/2}\right)^2 + (2e^{2t})^2}$$

$$= \sqrt{t^{-2} + t^2(t^2+1)^{-1} + 4e^{4t}}$$

acceleration:

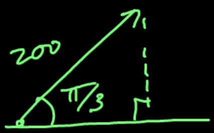
$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$

$$= \left\langle t^{-2}, t\left(-\frac{1}{2}(t^2+1)^{-3/2}\cdot 2t\right) + (t^2+1)^{-1/2}, 4e^{2t} \right\rangle$$

$$= \left\langle t^{-2}, (t^2+1)^{-1/2} - t^2(t^2+1)^{-3/2}, 4e^{2t} \right\rangle$$

Ex: A projectile is fired with initial speed 200 m/s at an angle of elevation 60° above the horizontal. Find...

Sol: $|\vec{v}(0)| = 200$, $\vec{r}(0) = \vec{0}$ $\therefore \vec{v}(0) = 100\langle 1, \sqrt{3}\rangle$





$\vec{a}(t) = -9.8\vec{j}$

$\vec{v}(t) = \int \vec{a}(t)\,dt = \vec{C} - 9.8t\,\vec{j}$

$\vec{v}(t) = \langle 100, 100\sqrt{3} - 9.8t\rangle$

ⓐ the range of the projectile. $\vec{r}(t) = \int \vec{v}(t)\,dt = \langle 100t, 100\sqrt{3}t - 4.9t^2\rangle + \vec{D}$

$\therefore \vec{r}(t) = \langle 100t, 100\sqrt{3}t - 4.9t^2\rangle$

$\vec{r}(t)$ has a 0 y-component

iff $100\sqrt{3}\,t - 4.9t^2 = 0 \iff t = 0$ OR $100\sqrt{3} - 4.9t = 0$

$\iff t = 0$ OR $t = \frac{1000\sqrt{3}}{49}$

$\therefore$ the range of the projectile is $100 \cdot \frac{1000\sqrt{3}}{49} = \frac{100\,000\sqrt{3}}{49}$ meters.

ⓑ the maximum height reached by the projectile.

$\vec{r}(t) = \langle 100t, 100\sqrt{3}t - 4.9t^2\rangle$

The max height will be reached when

$y'(t) = 0$ i.e. $100\sqrt{3} - 9.8t = 0 \iff t = \frac{100\sqrt{3}}{9.8} = \frac{500\sqrt{3}}{49}$

$\therefore$ the max height is

$y\left(\frac{500\sqrt{3}}{49}\right) = \frac{50000\cdot 3}{49} - \frac{250000\cdot 3}{49^2}\cdot\frac{49}{10} = \frac{3}{49}\left(50000 - 25000\right)$

$= \frac{75000}{49}$ meters

ⓒ the speed at which the projectile hits the ground.

$\vec{v}(t) = \langle 100, 100\sqrt{3} - 9.8t\rangle$

$\therefore$ speed is $|\vec{v}(t)| = \sqrt{100^2 + \left(100\sqrt{3} - \frac{49}{5}t\right)^2}$

$\therefore$ speed upon impact is precisely

$\left|\vec{v}\left(\frac{1000\sqrt{3}}{49}\right)\right| = \sqrt{100^2 + \left(100\sqrt{3} - 200\sqrt{3}\right)^2}$

$= \sqrt{100^2 + 100^2\cdot 3} = 100\sqrt{1+3} = 200$ m/s

TB

Ex: The position of a particle is given by $\vec{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$.
At what time is the particle's speed minimized?

Sol: $|\vec{v}(t)| = \sqrt{\vec{v}(t) \cdot \vec{v}(t)}$

$|\vec{v}(t)| = |\vec{r}\,'(t)|$ $\qquad \vec{r}\,'(t) = \langle 2t, 5, 2t-16 \rangle$

$f(t) = \sqrt{(2t)^2 + 5^2 + (2t-16)^2} = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} = \sqrt{8t^2 - 64t + 281}$ ← want to minimize!

$f'(t) = \frac{1}{2}(8t^2 - 64t + 281)^{-1/2} \cdot (16t - 64) = \dfrac{8(t-4)}{\sqrt{8t^2 - 64t + 281}}$ ←

Critical points: $t = 4$, sols to $8t^2 - 64t + 281 = 0$

i.e. $t = \dfrac{64 \pm \sqrt{64^2 - 4 \cdot 8 \cdot 281}}{2 \cdot 8} = 4 \pm \dfrac{4\sqrt{256 - 562}}{16}$ — no real solutions! reject.

f': $-$ ↘ 0 ↗ $+$ at 4

$\therefore$ we obtain a minimum at $t = 4$. $\boxed{\checkmark}$

NB: $\vec{v}(t) \cdot \vec{v}(t) \geq 0$
and $\sqrt{x}$ is strictly increasing.

$\therefore$ $|\vec{v}(t)|$ is minimized precisely when $\vec{v}(t) \cdot \vec{v}(t)$ is minimized.

$f(t) = \vec{v}(t) \cdot \vec{v}(t) = 8t^2 - 64t + 281$, so $f'(t) = 16t - 64$

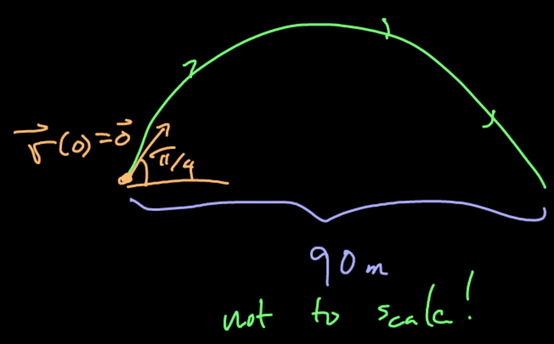
has $f'(t) = 0 \iff t = 4$ $\quad$ use first derivative test:

f': ↘ 0 ↗ at $t = 4$

$\therefore$ the speed is minimized at $t = 4$. $\boxed{\checkmark}$

Ex: A ball is thrown at an angle of $45°$ above the horizontal.
The ball lands $90\,m$ away. What is the ball's initial speed?

Sol: $\vec{a}(t) = -9.8\vec{j}$ $\qquad \vec{v}(0) = ?$ ← want this



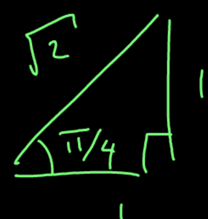
$\vec{v}(t) = \int \vec{a}(t)\,dt = \vec{c} - 9.8t\vec{j}$

$\vec{r}(t) = \int \vec{v}(t)\,dt = \vec{D} + \vec{c}t - 4.9t^2\vec{j}$

$= \langle D_1 + c_1 t, \; D_2 + c_2 t - 4.9t^2 \rangle$

when the ball lands: $D_1 + c_1 t_0 = 90$ ←

$\vec{v}(0) = \alpha\langle 1, 1 \rangle \implies c_1 = \alpha = c_2$



$\therefore$ $t_0 = \dfrac{90}{c_1}$, hence $y(t_0) = 0$ yields

$0 = c_1 t_0 - 4.9t_0^2 = c_1 \cdot \dfrac{90}{c_1} - 4.9 \cdot \left(\dfrac{90}{c_1}\right)^2 = 90\left(1 - \dfrac{441}{c_1^2}\right)$

$\therefore$ $0 = c_1^2 - 441$ $\quad$ hence $c_1 = \pm\sqrt{441}$ ← reject $-\sqrt{441}$

Hence $\vec{v}(0) = \langle c_1, c_1 \rangle = \sqrt{441}\,\langle 1, 1 \rangle$. $\boxed{\checkmark}$